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An algebraic construction of generalized coherent states associated with *q*-deformed models for primary shape-invariant systems

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Abstract
Generalized coherent states for primary shape invariant potential systems quantum deformed by different models are constructed using an algebraic approach based on supersymmetric quantum mechanics. We show that this generalized formalism is able (a) to supply the essential requirements necessary to establish a connection between classical and quantum formulations of a given system, (b) to reproduce, as particular cases, results already known for shape-invariant systems (such as standard harmonic oscillator and Pöschl–Teller potentials as well as quantum deformed harmonic oscillator models) and (c) point to a formalism that provides a unified description of the different kind of coherent states for quantum systems, deformed or not deformed.

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1. Introduction

Coherent states were first introduced by Schrödinger [1], who was interested in finding quantum-mechanics states which provide a close connection between quantum and classical formulations of a given physical system. Based on the Heisenberg–Weyl group and applied specifically to the harmonic oscillator system, the original coherent state introduced by Schrödinger has been extended to a large number of Lie groups with square integrable representation [2]. Today these extensions represent many applications in a number of fields of quantum theory, and especially in quantum optics and radiophysics. In particular, they are used as bases of coherent states path integrals [3] or dynamical wavepackets for describing the quantum systems in semi-classical approximations [4]. There are different definitions of coherent states. The first one, often called Barut–Girardello coherent states [5], assumes that the coherent states are eigenstates with complex eigenvalues of an annihilation group operator. The second definition, often called Perelomov coherent states [6], assumes the existence of an unitary z-displacement operator whose action on the ground state of the system gives the coherent state parameterized by z, with $z \in \mathbb{C}$. The last definition, based on the Heisenberg uncertainty relation, often called *intelligent* coherent states [7], assumes that the coherent state gives the minimum-uncertainty value $\Delta x \Delta p = \frac{\hbar}{2}$, and maintains this relation in time because of its temporal stability. These three different definitions are equivalent only in the special case of the Heisenberg–Weyl group, the dynamical symmetry group of the harmonic oscillator.

A class of generalized coherent states which has evoked a lot of interest is connected with deformed harmonic oscillator algebras. The development of quantum groups and quantum algebras motivated great interest in q-deformed algebraic structures, and in particular in the q-harmonic oscillators. Until now quantum groups have found applications in solid-state physics [8], nuclear physics [9, 10], quantum optics [11] and conformal field theories [12]. Quantum algebras are deformed versions of the usual Lie algebras obtained by introducing a deformation parameter q. In this sense the quantum algebras provide us with a class of symmetries which is richer than the usual class of Lie symmetries; the latter is contained in the former as a special case (when $q \to 1$). Therefore the quantum algebras may turn out to be appropriate tools for describing symmetries of physical systems which cannot be described by ordinary Lie algebras.

In a parallel development, the extension of coherent states for systems other than the harmonic oscillator has attracted much attention for the past several years [13–17]. There are a large number of different approaches to this problem and the one to be presented here is based on the supersymmetric quantum mechanics. Supersymmetric quantum mechanics [18] deals with pairs of Hamiltonians $\hat{H}$ and $\hat{H}'$ which have the same energy spectra, but different eigenstates. A number of such pairs of Hamiltonians share an integrability condition called shape invariance [19]. Although not all exactly solvable problems are shape-invariant, shape invariance, especially in its algebraic formulation [16, 20, 21], is a powerful technique to study exactly solvable systems.

In earlier papers, by using an algebraic approach, we introduced coherent states for self-similar potentials [16], a class of shape-invariant systems, and presented a possible generalization of these coherent-states and its relation with Ramanujan's integrals [17]. After that we extended this generalized formalism to all shape-invariant systems in [22]. Later we introduced a quantum deformed theory [23] applicable to all shape-invariant systems by defining q-deformed ladder operators that satisfy q-deformed commutation relations. The purpose of the present paper is to build generalized coherent states for the quantum deformed models obtained from the primary shape-invariant systems in our previous paper [23] and show that these generalized coherent states satisfy the essential principles embodied in Schrödinger's original idea. It is worth noting that until now the studies involving the extension of coherent states for q-deformed systems were restricted to the harmonic oscillator models. In this sense the plan of this paper is as follows: for the sake of completeness we will briefly review the fundamentals principles of the algebraic formulation to shape invariance in section 2 and the basic facts of the algebraic q-deformed theory for shape-invariant systems in section 3. In section 4 we introduce the fundamentals principles of our generalized coherent states and its basic properties. We apply our general formalism and work out some possible examples of coherent states to q-deformed primary shape-invariant systems in section 5 and show that the known results found in the literature for the q-deformed harmonic oscillator can be obtained with particular cases of our generalized expression. Finally, conclusion and brief remarks close the paper in section 6.

2. Algebraic formulation to shape invariance

Supersymmetric quantum mechanics is generally studied in the context of one-dimensional systems. The partner Hamiltonians $\hat{H} = -\frac{\hbar^2}{2M}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_-(x) = \hbar\Omega\hat{A}^\dagger\hat{A}$ and $\hat{H}' = -\frac{\hbar^2}{2M}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_+(x) = \hbar\Omega\hat{A}\hat{A}^\dagger$ can be written in terms of the dimensionless operators $\hat{A} \equiv \{W(x) + \mathrm{i}\hat{p}/\sqrt{2M}\}/\sqrt{\hbar\Omega}$ and $\hat{A}^\dagger \equiv \{W(x) - \mathrm{i}\hat{p}/\sqrt{2M}\}/\sqrt{\hbar\Omega}$, where $\hbar\Omega$ is a constant energy scale factor and $W(x)$ is the superpotential which is related to the partner potentials via $V_\pm(x) = W^2(x) \pm \frac{\hbar}{\sqrt{2M}}\frac{\mathrm{d}W(x)}{\mathrm{d}x}$. The Hamiltonian $\hat{H}$ is called shape-invariant if the condition $\hat{A}(a_1)\hat{A}^\dagger(a_1) = \hat{A}^\dagger(a_2)\hat{A}(a_2) + R(a_1)$ is satisfied [19]. The parameter a_2 of the Hamiltonian is a function of its parameter a_1 and the remainder $R(a_1)$ is independent of the dynamical variables such as position and momentum. In the cases studied so far, the parameters a_1 and a_2 are related by either a translation [20, 24] or a scaling [16, 17, 25]. Introducing the similarity transformation $\hat{T}(a_1)\hat{O}(a_1)\hat{T}^\dagger(a_1) = \hat{O}(a_2)$ that replace a_1 with a_2 in a given operator and the operators

$$\hat{B}_+ = \hat{A}^\dagger(a_1)\hat{T}(a_1) \qquad \text{and} \qquad \hat{B}_- = \hat{B}_+^\dagger = \hat{T}^\dagger(a_1)\hat{A}(a_1), \tag{1}$$

the partner Hamiltonians take the forms $\hat{H} = \hbar\Omega\hat{\mathcal{H}}_-$ and $\hat{H}' = \hbar\Omega\hat{T}\hat{\mathcal{H}}_+\hat{T}^\dagger$, where $\hat{\mathcal{H}}_\pm = \hat{B}_\mp\hat{B}_\pm$, and the condition of shape invariance can be written as the commutation relation

$$[\hat{B}_-, \hat{B}_+] = \hat{T}^\dagger(a_1)R(a_1)\hat{T}(a_1) \equiv R(a_0) \tag{2}$$

where we used the identity $R(a_n) = \hat{T}(a_1)R(a_{n-1})\hat{T}^\dagger(a_1)$ valid for any $n \in \mathbb{Z}$. The commutation relation (2) suggests that $\hat{B}_-$ and $\hat{B}_+$ are the appropriate creation and annihilation operators for the spectra of the shape-invariant potentials provided that their non-commutativity with $R(a_1)$ is taken into account. The additional relations

$$R(a_n)\hat{B}_+ = \hat{B}_+R(a_{n-1}) \qquad \text{and} \qquad R(a_n)\hat{B}_- = \hat{B}_-R(a_{n+1}) \tag{3}$$

readily follow from these results.

The ground state of the Hamiltonian $\hat{\mathcal{H}}_-$ satisfies the condition $\hat{A}|\Psi_0\rangle = 0 = \hat{B}_-|\Psi_0\rangle$ and using the relations above it is possible to show that its nth excited eigenstate

$$\hat{\mathcal{H}}_-|\Psi_n\rangle = e_n|\Psi_n\rangle \qquad \text{and} \qquad \hat{\mathcal{H}}_+|\Psi_n\rangle = \{e_n + R(a_0)\}|\Psi_n\rangle \tag{4}$$

can be written in a normalized form as

$$|\Psi_n\rangle = \frac{1}{\sqrt{R(a_1) + R(a_2) + \cdots + R(a_n)}}\hat{B}_+ \cdots \frac{1}{\sqrt{R(a_1) + R(a_2)}}\hat{B}_+\frac{1}{\sqrt{R(a_1)}}\hat{B}_+|\Psi_0\rangle \tag{5}$$

with the related eigenvalues e_n given by $e_0 = 0$ and

$$e_n = \sum_{k=1}^{n} R(a_k), \qquad \text{for} \quad n \geqslant 1. \tag{6}$$

3. Quantum deformation of shape-invariant systems

The studies of the quantum algebras are concentrated mainly to the harmonic oscillator quantum systems, the simplest shape-invariant potential. In a previous study [23] we developed several quantum deformed models applicable to all shape-invariant potential systems. In that paper, introducing q-deformed forms for the primary creation and annihilation operators $\hat{B}_\pm$, we define new q-deformed ladder operators and obtain their q-commutations relations. In this section we briefly illustrate this procedure. Further details are given in [23].

3.1. Generalized standard q-deformed model

The q-deformed forms for the creation and annihilation operators can be defined as

$$\hat{B}_{\pm}^{(q)} \equiv \left\{\hat{B}_{\mp}^{(q)}\right\}^{\dagger} = \hat{B}_{\pm}\sqrt{\frac{[\hat{B}_{\mp}\hat{B}_{\pm}]_q}{\hat{B}_{\mp}\hat{B}_{\pm}}} = \sqrt{\frac{[\hat{B}_{\pm}\hat{B}_{\mp}]_q}{\hat{B}_{\pm}\hat{B}_{\mp}}}\,\hat{B}_{\pm}, \tag{7}$$

where we used the q-operators extension of the q-numbers definition $[x]_q \equiv (q^x - q^{-x})/(q - q^{-1})$ and took into account that for any analytical function $f(x)$ the property $\hat{B}_{\pm}f(\hat{B}_{\mp}\hat{B}_{\pm}) = f(\hat{B}_{\pm}\hat{B}_{\mp})B_{\pm}$ is valid. Since $\lim_{q\to 1}[x]_q = x$, in this limit the postulated relations (7) tend to the usual generalized ladder operators (1). With the q-deformed ladder operators we can write down the q-deformed form of the Hamiltonian $\hat{H}$ as

$$\hat{H}_B^{(q)} = \hbar\Omega\hat{B}_+^{(q)}\hat{B}_-^{(q)} = \hbar\Omega[\hat{B}_+\hat{B}_-]_q \qquad \text{and} \qquad \hat{B}_-^{(q)}\hat{B}_+^{(q)} = [\hat{B}_-\hat{B}_+]_q. \tag{8}$$

Since $[\hat{H}_B^{(q)}, \hat{H}] = 0$, these Hamiltonians have the common set of eigenstates $|\Psi_n\rangle$ give by equation (5). Taking into account equations (4), (7) and (8), we can show that

$$\hat{H}_B^{(q)}|\Psi_n\rangle = \hbar\Omega[e_n]_q|\Psi_n\rangle \qquad \text{and} \qquad \hat{B}_-^{(q)}\hat{B}_+^{(q)}|\Psi_n\rangle = [e_n + R(a_0)]_q|\Psi_n\rangle. \tag{9}$$

Using equations (8) and relation (2) we can obtain the q-commutation relation between the $\hat{B}_{\pm}^{(q)}$ operators

$$\hat{B}_-^{(q)}\hat{B}_+^{(q)} - q^{\pm R(a_0)}\hat{B}_+^{(q)}\hat{B}_-^{(q)} = [R(a_0)]_q q^{\mp\hat{B}_+\hat{B}_-}. \tag{10}$$

3.2. Generalized standard Q-deformed models

As a second way to construct a q-deformed model for a shape-invariant potential we define the operators

$$\hat{C}_{\pm}^{(q)} = \{\hat{C}_{\mp}^{(q)}\}^{\dagger} = \frac{1}{\sqrt{q}}q^{\frac{1}{2}(\hat{B}_{\pm}\hat{B}_{\mp})}\hat{B}_{\pm}^{(q)} = \frac{1}{\sqrt{q}}\hat{B}_{\pm}^{(q)}q^{\frac{1}{2}(\hat{B}_{\mp}\hat{B}_{\pm})}. \tag{11}$$

Using the results of equations (8), relation (2) and the commutation between any function of the remainders $R(a_n)$ and the couple of operators $\hat{B}_{\pm}\hat{B}_{\mp}$, it is possible to establish the q-deformed commutation relation

$$\hat{C}_-^{(q)}\hat{C}_+^{(q)} - q^{2R(a_0)}\hat{C}_+^{(q)}\hat{C}_-^{(q)} = q^{R(a_0)}[R(a_0)]_q/q. \tag{12}$$

There is an alternative quantum numbers definition called the Q-numbers and given by $[x]_Q = (Q^x - 1)/(Q - 1)$. Using the Q-operators generalization of this definition and changing $q^2 \to Q$, it is possible to rewrite (11) as

$$\hat{C}_{\pm}^{(q)} \equiv \hat{B}_{\pm}^{(Q)} = \hat{B}_{\pm}\sqrt{\frac{[\hat{B}_{\mp}\hat{B}_{\pm}]_Q}{\hat{B}_{\mp}\hat{B}_{\pm}}} = \sqrt{\frac{[\hat{B}_{\pm}\hat{B}_{\mp}]_Q}{\hat{B}_{\pm}\hat{B}_{\mp}}}\,\hat{B}_{\pm} \tag{13}$$

and show that the q-deformed commutation relation (12) can be rewritten in a Q-deformed version as

$$\hat{B}_-^{(Q)}\hat{B}_+^{(Q)} - Q^{R(a_0)}\hat{B}_+^{(Q)}\hat{B}_-^{(Q)} = [R(a_0)]_Q. \tag{14}$$

With these definitions for quantum deformed ladder operators we can write down the Q-deformed form of $\hat{H}$ as

$$\hat{H}_C^{(q)} = \hbar\Omega\hat{C}_+^{(q)}\hat{C}_-^{(q)} = \hbar\Omega\hat{B}_+^{(Q)}\hat{B}_-^{(Q)} = \hbar\Omega[\hat{B}_+\hat{B}_-]_Q \qquad \text{and} \qquad \hat{B}_-^{(Q)}\hat{B}_+^{(Q)} = [\hat{B}_-\hat{B}_+]_Q. \tag{15}$$

Taking into account equations (4), (7), (9), (11), (15) and the commutation relation $[\hat{H}_C^{(q)}, \hat{H}] = 0$, we conclude that these Hamiltonians have the common set of eigenstates (5) which satisfy the eigenvalue equation

$$\hat{H}_C^{(q)}|\Psi_n\rangle = \hbar\Omega[e_n]_Q|\Psi_n\rangle \qquad \text{and} \qquad \hat{B}_-^{(Q)}\hat{B}_+^{(Q)}|\Psi_n\rangle = [e_n + R(a_0)]_Q|\Psi_n\rangle. \tag{16}$$

3.3. Generalized maths-type q-deformed model

Another q-deformed model can be obtained introducing the operators

$$\hat{D}_-^{(q)} = q^{-\frac{1}{2}R(a_0)}\hat{B}_-^{(q)}q^{\frac{1}{2}(\hat{B}_+\hat{B}_-)} = q^{-\frac{1}{2}R(a_0)}q^{\frac{1}{2}(\hat{B}_-\hat{B}_+)}\hat{B}_-^{(q)} \tag{17}$$

$$\hat{D}_+^{(q)} = \{\hat{D}_-^{(q)}\}^\dagger = q^{\frac{1}{2}(\hat{B}_+\hat{B}_-)}\hat{B}_+^{(q)}q^{-\frac{1}{2}R(a_0)} = \hat{B}_+^{(q)}q^{\frac{1}{2}(\hat{B}_-\hat{B}_+)}q^{-\frac{1}{2}R(a_0)}. \tag{18}$$

Using this definition and the results of equations (8) and (2) we establish the q-deformed commutation relation

$$\hat{D}_-^{(q)}\hat{D}_+^{(q)} - q^{[R(a_0)+R(a_1)]}\hat{D}_+^{(q)}\hat{D}_-^{(q)} = [R(a_0)]_q. \tag{19}$$

On the other hand, with the operators $\hat{D}_\pm^{(q)}$ we can write down the correspondent q-deformed form of $\hat{H}$ as

$$\begin{aligned} &\hat{H}_D^{(q)} = \hbar\Omega\hat{D}_+^{(q)}\hat{D}_-^{(q)} = \hbar\Omega q^{-R(a_1)}q^{(\hat{B}_+\hat{B}_-)}[\hat{B}_+\hat{B}_-]_q \qquad \text{and} \\ &\hat{D}_-^{(q)}\hat{D}_+^{(q)} = q^{-R(a_0)}q^{(\hat{B}_-\hat{B}_+)}[\hat{B}_-\hat{B}_+]_q. \end{aligned} \tag{20}$$

Since $\left[\hat{H}_D^{(q)}, \hat{H}\right] = 0$, the common set of eigenstates (5) for these Hamiltonians satisfy the eigenvalue equation

$$\hat{H}_D^{(q)}|\Psi_n\rangle = \hbar\Omega q^{(e_n-e_1)}[e_n]_q|\Psi_n\rangle \qquad \text{and} \qquad \hat{D}_-^{(q)}\hat{D}_+^{(q)}|\Psi_n\rangle = q^{e_n}[e_n + R(a_0)]_q|\Psi_n\rangle. \tag{21}$$

3.4. Quantum deformed shape-invariant systems

As shown in [23], the three previous quantum deformed models correspond to the shape-invariant generalization of the standard Arik and Coon models [26], first introduced for the harmonic oscillator potential systems. The q-deformed commutation relations (10), (12) and (19) between the q-deformed ladder operators defined in each model show that the shape invariance of the primary system, represented by the commutation relation (2), is broken after the quantum deformation process. This shape invariance property is only recovered when we take the limit of $q \to 1$. Obviously it is a consequence of the basic assumptions used to build the quantum-deformed models. However, taking advantage of the freedom permitted in general for a primary shape-invariant system, it is possible to construct a q-deformed model which, unlike the previous ones, preserves the shape invariance of the primary system after the quantum deformation process. In this sense, we introduce the operators

$$\begin{aligned} &\hat{S}_-^{(q)} = \mathcal{F}_q\hat{B}_-^{(q)}q^{\frac{1}{2}(\hat{B}_+\hat{B}_-)} = \mathcal{F}_q q^{\frac{1}{2}(\hat{B}_-\hat{B}_+)}\hat{B}_-^{(q)} \qquad \text{and} \\ &\hat{S}_+^{(q)} = \{\hat{S}_-^{(q)}\}^\dagger = q^{\frac{1}{2}(\hat{B}_+\hat{B}_-)}\hat{B}_+^{(q)}\mathcal{F}_q = \hat{B}_+^{(q)}q^{\frac{1}{2}(\hat{B}_-\hat{B}_+)}\mathcal{F}_q \end{aligned} \tag{22}$$

where $\mathcal{F}_q \equiv \mathcal{F}(q; a_0, a_1, a_2, \ldots) \in \mathbb{R}$. Observe that the Hermitian conjugation condition of $\hat{S}_\pm^{(q)}$ imply that $q \in \mathbb{R}$. Using these definitions, the commutator (2) and equations (8) and (3) it is possible get the commutator

$$[\hat{S}_-^{(q)}, \hat{S}_+^{(q)}] = \mathcal{G}_0, \qquad \text{with} \qquad \mathcal{G}_0 \equiv \mathcal{F}_q^2 q^{R(a_0)}[R(a_0)]_q \tag{23}$$

since we assume that the arbitrary functional $\mathcal{F}_q$ satisfies the constraint

$$\hat{T}(a_1)\mathcal{F}_q^2\hat{T}^\dagger(a_1) = q^{2R(a_0)}\mathcal{F}_q^2. \tag{24}$$

Comparing equations (2) and (23) we conclude that the latter can be associated with a shape invariance condition as the former and that $\hat{S}_-^{(q)}$ and $\hat{S}_+^{(q)}$ are the appropriate creation and annihilation operators for the spectra of the q-deformed shape-invariant systems whose Hamiltonian is given by

$$\hat{H}_S^{(q)} = \hbar\Omega\hat{S}_+^{(q)}\hat{S}_-^{(q)} = \hbar\Omega q^{2R(a_0)}\mathcal{F}_q^2 q^{(\hat{B}_+\hat{B}_-)}[\hat{B}_+\hat{B}_-]_q. \tag{25}$$

With these definitions, relations (3) and equation (23) we can write down the additional commutation relations

$$\begin{aligned} \left[\hat{H}_S^{(q)}, \left(\hat{S}_+^{(q)}\right)^n\right] &= +\hbar\Omega\{\mathcal{G}_1 + \mathcal{G}_2 + \cdots + \mathcal{G}_n\}\left(\hat{S}_+^{(q)}\right)^n, \\ \left[\hat{H}_S^{(q)}, \left(\hat{S}_-^{(q)}\right)^n\right] &= -\hbar\Omega\left(\hat{S}_-^{(q)}\right)^n\{\mathcal{G}_1 + \mathcal{G}_2 + \cdots + \mathcal{G}_n\} \end{aligned} \tag{26}$$

where $\mathcal{G}_n = \hat{T}(a_1)\mathcal{G}_{n-1}\hat{T}^\dagger(a_1)$. Using that $\hat{B}_-|\Psi_0\rangle = 0$ and equations (7) and (22) we can also show that $\hat{S}_-^{(q)}|\Psi_0\rangle = 0$. From this result and the commutator (26) it follows that

$$\hat{H}_S^{(q)}\left\{\left(\hat{S}_+^{(q)}\right)^n|\Psi_0\rangle\right\} = \hbar\Omega\{\mathcal{G}_1 + \mathcal{G}_2 + \cdots + \mathcal{G}_n\}\left\{\left(\hat{S}_+^{(q)}\right)^n|\Psi_0\rangle\right\}, \tag{27}$$

i.e., $\left|\Psi_n^{(S)}\right\rangle \equiv \left(\hat{S}_+^{(q)}\right)^n|\Psi_0\rangle$ is an eigenstate of the Hamiltonian $\hat{H}_S^{(q)}$ with the eigenvalue

$$E_n^{(S)} = \hbar\Omega\sum_{k=1}^{n}\mathcal{G}_k = \hbar\Omega q^{2R(a_0)}\mathcal{F}_q^2 q^{e_n}[e_n]_q. \tag{28}$$

4. Construction of generalized coherent states for *q*-deformed shape-invariant systems

Annihilation-operator coherent states for shape-invariant potentials were introduced in [14, 16]. Here we follow the notation of [16]. Our first step is to introduce the necessary tools to be used in this construction. After we obtain the coherent state we must verify if this state satisfies the set of four essential requirements, introduced and discussed in [22, 27], necessary for a close connection between classical and quantum formulations of a given system: (a) label continuity; (b) overcompleteness or resolution of unity; (c) temporal stability; and (d) action identity. The first two requirements are standard and rely on the algebraic structure behind the system in question, while the last two are more general and relate to the classical connection question. To work with a general formulation that can include the four different q-deformations theories presented above, let us introduce the general q-deformed Hamiltonian defined by $\hat{H}_X^{(q)} = \hbar\Omega\hat{\mathcal{H}}_X^{(q)}$ with $\hat{\mathcal{H}}_X^{(q)} = \hat{X}_+^{(q)}\hat{X}_-^{(q)}$, where the generalized ladder operators can represent $\hat{X}_\pm^{(q)} \equiv \hat{B}_\pm^{(q)}, \hat{C}_\pm^{(q)}, \hat{D}_\pm^{(q)}\text{or}\hat{S}_\pm^{(q)}$, and consequently $\hat{H}_X^{(q)} \equiv \hat{H}_B^{(q)}, \hat{H}_C^{(q)}, \hat{H}_D^{(q)}\text{or}\hat{H}_S^{(q)}$. Obviously in this case, taking into account the results for each q-deformed model, we must have that

$$\hat{\mathcal{H}}_X^{(q)}|\Psi_n\rangle \equiv \hat{X}_+^{(q)}\hat{X}_-^{(q)}|\Psi_n\rangle = \varepsilon_n^{(X)}|\Psi_n\rangle, \qquad \text{where}$$

$$\varepsilon_n^{(X)} = \begin{cases} [e_n]_q\,, & X \equiv B; \\ [e_n]_Q\,, & X \equiv C; \\ q^{(e_n - e_1)}\,[e_n]_q\,, & X \equiv D; \\ q^{2R(a_0)}\mathcal{F}_q^2 q^{e_n}\,[e_n]_q\,, & X \equiv S, \end{cases} \tag{29}$$

e_n being given by equation (6).

4.1. Construction

Taking into account that $\hat{B}_-|\Psi_0\rangle = 0$ and looking at definitions (7), (11), (17) and (22), we conclude that the operator $\hat{X}_-^{(q)}$ does not have a left inverse in the Hilbert space of the eigenstates of the Hamiltonian $\hat{H}_X^{(q)}$. However, a right inverse for $\hat{X}_-^{(q)}$, $[\hat{X}_-^{(q)}\{\hat{X}_-^{(q)}\}^{-1} = \hat{1}]$, can be defined. Similarly the inverse of $\hat{\mathcal{H}}_X^{(q)}$ does not exist, but

$$\{\hat{\mathcal{H}}_X^{(q)}\}^{-1}\hat{X}_+^{(q)} = \{\hat{X}_-^{(q)}\}^{-1} \tag{30}$$

does. Therefore, if we define the Hermitian conjugate operators $\hat{K}_X^{(q)} = \hat{X}_-^{(q)}\{\hat{\mathcal{H}}_X^{(q)}\}^{-1/2}$ and $\{\hat{K}_X^{(q)}\}^\dagger = \{\hat{\mathcal{H}}_X^{(q)}\}^{-1/2}\hat{X}_+^{(q)}$, we can easily show that

$$\{\hat{X}_-^{(q)}\}^{-1} = \{\hat{\mathcal{H}}_X^{(q)}\}^{-1/2}\{\hat{K}_X^{(q)}\}^\dagger, \tag{31}$$

and the normalized form of the nth excited state of $\hat{H}$ and $\hat{H}_X^{(q)}$, given by (5), can be rewritten as

$$|\Psi_n\rangle = \left(\{\hat{K}_X^{(q)}\}^\dagger\right)^n|\Psi_0\rangle. \tag{32}$$

Then, taking into account equations (6), (29), (31) and (32) we can prove that $\{\hat{X}_-^{(q)}\}^{-n}|\Psi_0\rangle = C_n^{(X)}|\Psi_n\rangle$, where

$$\frac{1}{C_n^{(X)}} = \prod_{k=0}^{n-1}\sqrt{\Theta_{nk}^{(X)}}, \qquad \text{with} \quad \Theta_{nk}^{(X)} = \begin{cases} [e_n - e_k]_q, & X \equiv B; \\ [e_n - e_k]_Q, & X \equiv C; \\ q^{(e_n - e_{k+1})}[e_n - e_k]_q, & X \equiv D; \\ \{q^{R(a_0)}\mathcal{F}_q\}^2 q^{(e_n+e_k)}[e_n - e_k]_q, & X \equiv S. \end{cases} \tag{33}$$

Now we can define the generalized expression for the coherent state of the q-deformed shape-invariant systems as

$$|z; q; a_s\rangle_X = \sum_{n=0}^{\infty}\{z\mathcal{Z}_s^{(q)}(\hat{X}_-^{(q)})^{-1}\}^n|\Psi_0\rangle \qquad \text{with} \qquad z, \mathcal{Z}_s^{(q)} \in \mathbb{C}. \tag{34}$$

In this definition we used the shorthand notation $\mathcal{Z}_s^{(q)} \equiv \mathcal{Z}(q; a_1, a_2, a_3, \ldots)$ for an arbitrary functional, introduced to establish a more general approach. With relation (3) we can prove that $|z; q; a_s\rangle_X$ is an eigenstate of the q-deformed annihilation operator $\hat{X}_-^{(q)}$ since

$$\hat{X}_-^{(q)}|z; q; a_s\rangle_X = z\mathcal{Z}_{s-1}^{(q)}|z; q; a_s\rangle_X \qquad \text{where} \qquad \mathcal{Z}_{s-1}^{(q)} = \hat{T}^\dagger(a_1)\mathcal{Z}_s^{(q)}\hat{T}(a_1). \tag{35}$$

4.2. Normalization

Using the action of the $(\hat{X}_-^{(q)})^{-1}$ operator on the Hilbert space of the eigenstates $\{|\Psi_n\rangle, n = 0, 1, 2, \ldots\}$, and (3) we obtain the normalized Glauber's form [28] of the coherent state $|z; q; a_s\rangle_X$ as

$$\begin{aligned} |z; q; a_s\rangle_X &= \mathcal{N}_X(|z|^2; q; a_s)\sum_{n=0}^{\infty}\left\{\frac{z^n}{h_n^{(X)}(q; a_s)}\right\}|\Psi_n\rangle, \qquad \text{where} \\ \mathcal{N}_X(|z|^2; q; a_s) &= 1\Bigg/\sqrt{\sum_{n=0}^{\infty}\frac{|z|^{2n}}{|h_n^{(X)}(q; a_s)|^2}} \end{aligned} \tag{36}$$

and we used the shorthand notation $(a_s) \equiv [a_1, a_2, a_3, \ldots]$. The expansion coefficients are given by

$$h_0^{(X)}(q; a_s) = 1 \qquad \text{and} \qquad h_n^{(X)}(q; a_s) = \prod_{k=0}^{n-1}\left(\sqrt{\Theta_{nk}^{(X)}}\Big/\mathcal{Z}_{s+k}^{(q)}\right) \qquad \text{for} \quad n \geqslant 1 \tag{37}$$

with $\mathcal{Z}_{s+k}^{(q)} = \{\hat{T}(a_1)\}^k \mathcal{Z}_s^{(q)} \{\hat{T}^\dagger(a_1)\}^k$. It should be pointed out that the transformation properties between the potential parameters a_n, imposed by shape invariance condition, constrain the freedom in the definition of $\mathcal{Z}_s^{(q)}$. Besides that, when we consider relation (37), this potential parameter dependence in $\mathcal{Z}_s^{(q)}$ shows strong influence in the final expression of the expansion coefficient $h_n^{(X)}(q; a_s)$. Another aspect to detach about $\mathcal{Z}_s^{(q)}$ is its importance in the determination of the radius of convergence of the series which gives $\mathcal{N}_X(|z|^2; q; a_s)$ since this radius is given by $\mathcal{R} = \limsup_{n\to+\infty} \sqrt[n]{\left|h_n^{(X)}(q; a_s)\right|^2}$. To conclude, note that the coherent states $|z; q; a_s\rangle_X$ form an over-complete linearly dependent set since, although they can be normalized, we have ${}_X\langle z'; q; a_s|z; q; a_s\rangle_X = \mathcal{N}_X(|z'|^2; q; a_s)\mathcal{N}_X(|z|^2; q; a_s)\big/\mathcal{N}_X^2(zz'^*; q; a_s)$.

4.3. Continuity of labelling

From the continuity of the overlapping factor between two different coherent states we can show that the normalizable coherent state defined in (34) is continuous in the labels z and s. It means that if we have $(z, a_s) \longrightarrow (z', a_{s'})$ then we will find that $|z; q; a_s\rangle_X \longrightarrow |z'; q; a_{s'}\rangle_X$ or, in others words, if

$$\begin{cases} z - z' \to 0 \\ s - s' \to 0 \end{cases} \implies \quad \delta_z \equiv \left|\frac{1}{2}\{|z; q; a_s\rangle_X - |z'; q; a_{s'}\rangle_X\}\right|^2 \\ = 1 - \mathrm{Re}\{{}_X\langle z'; q; a_{s'}|z; q; a_s\rangle_X\} \to 0. \tag{38}$$

This condition can be easily verified if we use equation (36) to obtain

$${}_X\langle z'; q; a_{s'}|z; q; a_s\rangle_X = \mathcal{N}_X(|z'|^2; q; a_{s'})\mathcal{N}_X(|z|^2; q; a_s) \sum_{n=0}^{\infty} \left\{\frac{(z'z)^n}{h_n^{(X)*}(q; a_{s'})h_n^{(X)}(q; a_s)}\right\} \tag{39}$$

and consider that this result gives a continuous function.

4.4. Overcompleteness

We now investigate the completeness or resolution of unity of the generalized coherent states introduced by equation (34) by assuming the existence of a positive-definite weight function $w_X(|z|^2; q; a_s)$ so that

$$\int_{\mathbb{C}} \mathrm{d}^2 z\, w_X(|z|^2; q; a_s)|z; q; a_s\rangle_{X\,X}\langle z; q; a_s| = \hat{1} \tag{40}$$

where the integral is taken over the entire complex plane. Inserting equation (36) into equation (40) we find

$$\int_{\mathbb{C}} \mathrm{d}^2 z\, w_X(|z|^2; q; a_s)\mathcal{N}_X^2(|z|^2; q; a_s) \sum_{m,n=0}^{\infty} \left\{\frac{z^{*m} z^n}{h_m^{(X)*}(q; a_s)h_n^{(X)}(q; a_s)}\right\} |\Psi_n\rangle\langle\Psi_m| = \hat{1}. \tag{41}$$

Using the orthonormality of the eigenstates $|\Psi_n\rangle$ and the polar form $z \equiv r\mathrm{e}^{\mathrm{i}\phi}$, $\mathrm{d}^2 z = r\mathrm{d}r\,\mathrm{d}\phi$ in the diagonal matrix elements of (41), we can express the resolution of unity condition as

$$\int_0^\infty \mathrm{d}\rho\rho^n \mathcal{W}_X(\rho; q; a_s) = \left|h_n^{(X)}(q; a_s)\right|^2 \qquad \text{where} \\ \mathcal{W}_X(\rho; q; a_s) = \pi\mathcal{N}_X^2(\rho; q; a_s)w_X(\rho; q; a_s) \tag{42}$$

and ρ stands for r^2. In other words, equation (42) provides the set of moments $\{\rho_n\}$ of the distribution function $\mathcal{W}_X(\rho; q; a_s)$, since we assume all moments exist and have finite values. Therefore the problem of finding the measure $w_X(\rho; q; a_s)$ reduces to a Stieltjes or a Hausdorff power-moment distribution problem [29]. Note that explicit computation of $w_X(\rho; q; a_s)$ requires the knowledge of the spectrum $\{e_n; n = 1, 2, \ldots\}$ and the form of $\mathcal{Z}_s^{(q)}$.

4.5. Temporal stability

The time evolution of the generalized coherent state (36) can be obtained by $|z; q; a_s; t\rangle_X = \hat{U}_X^{(q)}(t, 0)|z; q; a_s; 0\rangle_X$, where the time evolution operator fulfils the differential equation $\mathrm{i}\hbar\frac{\partial}{\partial t}\hat{U}_X^{(q)}(t, 0) = \hat{H}_X^{(q)}\hat{U}_X^{(q)}(t, 0)$, with the initial condition $\hat{U}_X^{(q)}(0, 0) = \hat{\mathbb{1}}$. Thus, $|z; q; a_s; t\rangle_X = \exp\left\{-\mathrm{i}\hat{H}_X^{(q)}t/\hbar\right\}|z; q; a_s; 0\rangle_X$, and if we consider expansion (36) and the results of equations (4) in this expression, we find

$$|z; q; a_s; t\rangle_X = \mathcal{N}_X(|z|^2; q; a_s)\sum_{n=0}^{\infty}\left\{\frac{z^n}{h_n^{(X)}(q; a_s)}\right\}\exp\left\{-\mathrm{i}\Omega\varepsilon_n^{(X)}t\right\}|\Psi_n\rangle. \tag{43}$$

To establish the temporal stability of $|z; q; a_s; t\rangle_X$, we utilize the freedom in the choice of the functional $\mathcal{Z}_s^{(q)}$ to redefine the expansion coefficients as $\bar{h}_n^{(X)}(q; a_s) = h_n^{(X)}(q; a_s)\exp\left\{\mathrm{i}\alpha\varepsilon_n^{(X)}\right\}$, where α is a real constant, $\varepsilon_n^{(X)}$ is given by (29) and $h_n^{(X)}(q; a_s)$ still given by equation (37). In these conditions we rewrite the coherent state $|z; q; a_s; 0\rangle_X$ as

$$\begin{aligned} |z; q; a_s; 0\rangle_X \quad &\Longrightarrow \quad |z, \alpha; q; a_s; 0\rangle_X = \mathcal{N}_X(|z|^2; q; a_s) \\ &\times\sum_{n=0}^{\infty}\left\{\frac{z^n}{h_n^{(X)}(q; a_s)}\right\}\exp\left\{-\mathrm{i}\alpha\varepsilon_n^{(X)}\right\}|\Psi_n\rangle, \end{aligned} \tag{44}$$

and its time-evolved form as

$$\begin{aligned} |z, \alpha; q; a_s; t\rangle_X &= \mathcal{N}_X(|z|^2; q; a_s)\sum_{n=0}^{\infty}\left\{\frac{z^n}{h_n^{(X)}(q; a_s)}\right\}\exp\left\{-\mathrm{i}\left(\alpha + \Omega t\right)\varepsilon_n^{(X)}\right\}|\Psi_n\rangle \\ &\equiv |z, \alpha + \Omega t; q; a_s; 0\rangle_X, \end{aligned} \tag{45}$$

showing that the time evolution of any such coherent state remains within the family of coherent states.

4.6. Action identity

To verify that we take the conjugate of equation (35) and use the definition of the creation operator $\hat{X}_+^{(q)}$ to obtain ${}_X\langle z; q; a_s|\hat{X}_+^{(q)} = {}_X\langle z; q; a_s|z^*\mathcal{Z}_{s-1}^{(q)*}$. Therefore with this result, equation (35) and expression of $\hat{H}_X^{(q)}$ we can calculate the expectation value

$$\left\langle\hat{H}_X^{(q)}\right\rangle = \frac{{}_X\langle z; q; a_s|\hat{H}_X^{(q)}|z; q; a_s\rangle_X}{{}_X\langle z; q; a_s|z; q; a_s\rangle_X} = \hbar\Omega\left\{\frac{{}_X\langle z; q; a_s|\hat{X}_+^{(q)}\hat{X}_-^{(q)}|z; q; a_s\rangle_X}{{}_X\langle z; q; a_s|z; q; a_s\rangle_X}\right\} = \hbar\Omega\left|z\mathcal{Z}_{s-1}^{(q)}\right|^2. \tag{46}$$

Considering this result and defining a canonical action variable $J_q = \hbar\beta_s^{(q)*}\beta_s^{(q)}$, with $\beta_s^{(q)} = z\mathcal{Z}_{s-1}^{(q)}$, we can write $\left\langle\hat{H}_X^{(q)}\right\rangle = \nu J_q$, so that $\dot{\nu} = \partial\left\langle\hat{H}_X^{(q)}\right\rangle/\partial J_q = \Omega$ and thus $\nu = \Omega t + \nu_0$, as required for a couple of canonical conjugate action-angle variables. Note that the normalized form (36) of the coherent state $|z; q; a_s\rangle_q$ implies that we must redefine $\beta_s^{(q)} = z\mathcal{Z}_{s-1}^{(q)}\mathcal{N}_X(|z|^2; q; a_{s-1})/\mathcal{N}_X(|z|^2; q; a_s)$.

5. Examples of coherent states for shape-invariant systems *q*-Deformed

Using the definition presented in the previous section we now illustrate the concept of generalized coherent states for quantum deformed shape-invariant systems. In these applications we consider the four different quantum deformed models presented in section 3 applied for the harmonic oscillator and the Pöschl–Teller potential systems.

5.1. Quantum deformed harmonic oscillator coherent states

We begin with this system because it is the simplest and most studied among the shape-invariant potential systems. In this case the partner potentials $V_{\pm}(x)$ are obtained with the superpotentials $W(x, a_1) = \sqrt{\hbar\Omega}\,(\beta x + \delta)$, where β and δ are real constants, while the remainders in the shape invariance condition (2) are given by [18] $R(a_n) = \sqrt{\hbar/(2M\Omega)}(a_n + a_{n+1})$. Taking into account that the parameters for this potential are related by $a_1 = a_2 = \cdots = a_n = \beta$, then we conclude that the remainders can be written as $R(a_n) = \gamma$, with $\gamma = \sqrt{2\hbar/(M\Omega)}\beta$, and thus

$$e_n = \sum_{k=1}^{n} R(a_k) = n\gamma. \tag{47}$$

5.1.1. Standard q-deformed coherent states $(X = B)$. In this case with (47) in (33) we find $\Theta_{nk}^{(B)} = [\gamma(n-k)]_q$. Because of the constant values of the potential parameters for this shape-invariant potential we must have $\mathcal{Z}_s^{(q)} = \mathcal{Z}_q$, a constant, which can be written in terms of the q-parameter. Using this fact and $\Theta_{nk}^{(B)}$ in (37) we obtain

$$h_n^{(B)}(q; a_s) = \sqrt{\frac{q^n(1-q^2)^n(q^{2\gamma}; q^{2\gamma})_n}{q^{\frac{1}{2}\gamma n(n+1)}\,\mathcal{Z}_q^{2n}}} \tag{48}$$

where the $q^{2\gamma}$-shifted factorial $(q^{2\gamma}; q^{2\gamma})_n$ is defined as $(p; q^{\eta})_0 = 1$ and

$$(p; q^{\eta})_n = \prod_{s=0}^{n-1}(1 - pq^{s\eta}), \qquad \text{with} \qquad n = 1, 2, 3, \ldots. \tag{49}$$

Therefore using the expansion coefficient (48) in (36) we obtain the normalized coherent state

$$|z; q; a_s\rangle_B = \frac{1}{\sqrt{E_{q^{2\gamma}}^{(1/2)}(|\xi_q|^2)}} \sum_{n=0}^{\infty} \left\{ \frac{q^{\frac{1}{4}\gamma n^2}}{\sqrt{(q^{2\gamma}; q^{2\gamma})_n}} \right\} \xi_q^n |n\rangle \qquad \text{where}$$
$$\xi_q = \sqrt{\frac{q^{\frac{\gamma}{2}}(1-q^2)}{q}}\, z\mathcal{Z}_q \tag{50}$$

and

$$E_{q^{\mu}}^{(\alpha)}(z) = \sum_{n=0}^{\infty} \left\{ \frac{q^{\frac{1}{2}\mu\alpha n^2}}{(q^{\mu}; q^{\mu})_n} \right\} z^n \tag{51}$$

is the q^{μ} extension of the one-parameter family of q-exponential functions [30] with $\alpha, \mu \in \Re$ and $q^{\mu} < 1$. Note that we took into account that in this case $|\Psi_n\rangle \to |n\rangle$, an element of the Fock space $\{|n\rangle; n = 0, 1, 2, \ldots\}$.

To make clear the generality of our result several remarks are in order at this point.

• First, if we take the limit when q goes to unity of the expansion factor (48) we find

$$\lim_{q\to 1}\left\{h_n^{(B)}(q;a_s)\right\} = \sqrt{\frac{\gamma^n n!}{\mathcal{Z}_1^n}} \quad \text{yielding}$$
$$\lim_{q\to 1}|z;q;a_s\rangle_B = \exp\left\{-\left(\frac{\mathcal{Z}_1|z|}{\sqrt{2\gamma}}\right)^2\right\}\sum_{n=0}^{\infty}\frac{\left(\frac{\mathcal{Z}_1 z}{\sqrt{\gamma}}\right)^n}{\sqrt{n!}}|n\rangle, \tag{52}$$

which is the generalized expression obtained in [22] for the coherent state of a non-deformed harmonic oscillator. As pointed out there, if we redefine $z \to \mathcal{Z}_1 z/\sqrt{\gamma}$ we obtain the usual expressions by bosonic coherent state models [2].

• As a particular case of the generalized q-deformed coherent states (50) we can assume that $\gamma = 1$ and $\mathcal{Z}_q = 1$ and thus, using the q-exponential function $\exp_q(z)$ defined in [31], we obtain

$$|z;q;a_s\rangle_B \to |z;q\rangle = \frac{1}{\sqrt{\exp_q(|z|^2)}}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{[n]_q!}}|n\rangle, \qquad \text{where} \qquad \exp_q(z) = \sum_{n=0}^{\infty}\frac{z^n}{[n]_q!} \tag{53}$$

with $[n]_q! = [n]_q[n-1]_q\ldots,[2]_q[1]_q$. This result is the q-coherent state $|z;q\rangle$ first introduced by Biedenharn [32] and used by Bracken *et al* [33] in their study involving a q-analogue of boson operators and Bargmann space.

• On the other hand, with the choice of $\gamma = \frac{1}{2}$ and $\mathcal{Z}_q = \sqrt{\sqrt{q}/(1+q)}$ we get

$$|z;q;a_s\rangle_B \to |z\rangle_q = \frac{1}{\sqrt{\exp_{\sqrt{q}}(|z|^2)}}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{[n]_{\sqrt{q}}!}}|n\rangle, \qquad \text{where now}$$
$$\exp_{\sqrt{q}}(z) = \sum_{n=0}^{\infty}\frac{z^n}{[n]_{\sqrt{q}}!} \tag{54}$$

with $[n]_{\sqrt{q}}! = [n]_{\sqrt{q}}[n-1]_{\sqrt{q}}\ldots,[2]_{\sqrt{q}}[1]_{\sqrt{q}}$ and $[n]_{\sqrt{q}} \equiv \left(q^{\frac{n}{2}} - q^{-\frac{n}{2}}\right)/\left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right)$. Expression (54) reproduce the $|z\rangle_q$ coherent state defined by Gray and Nelson [31] in their paper about q-analogue coherent states for harmonic oscillator systems.

5.1.2. Standard Q-deformed coherent states $(X = C)$. In this case inserting (47) into (33) we get $\Theta_{nk}^{(C)} = [\gamma(n-k)]_Q$. In the same way, we must have $\mathcal{Z}_s^{(Q)} = \mathcal{Z}_Q$, a constant. Therefore using this fact and $\Theta_{nk}^{(C)}$ in (37) we can show that

$$h_n^{(C)}(Q;a_s) = \frac{\sqrt{(Q^\gamma;Q^\gamma)_n}}{\{\sqrt{1-Q}\mathcal{Z}_Q\}^n} \tag{55}$$

where the Q^γ-shifted factorial $(Q^\gamma;Q^\gamma)_n$ has the same definition presented in (49), just replacing $q \to Q$. The expression of the normalized coherent state obtained substituting (55) in (36) is

$$|z;Q;a_s\rangle_C = \frac{1}{\sqrt{\exp_{Q^\gamma}\left(|\xi_Q|^2\right)}}\sum_{n=0}^{\infty}\frac{\xi_Q^n}{\sqrt{(Q^\gamma;Q^\gamma)_n}}|n\rangle, \qquad \text{where}$$
$$\exp_{Q^\mu}(z) = \sum_{n=0}^{\infty}\frac{z^n}{(Q^\mu;Q^\mu)_n} \tag{56}$$

is the Q^μ extension of the q-exponential function [34] with $\mu \in \Re$ and $Q^\mu < 1$. In this case $\xi_Q = \sqrt{1-Q}z\mathcal{Z}_Q$.

Let us consider some remarks about this general result.

• As the previous example, it is easy to verify that the limit $Q \to 1$ of (56) reproduce the generalized expression obtained in [22] for the coherent state of a non-deformed harmonic oscillator system.

• Taking the simple choice $\mathcal{Z}_q = \gamma = 1$ and using the relation $[n]_Q! = (Q; Q)_n/(1-Q)^n$, we obtain

$$|z; Q; a_s\rangle_C \to |z\rangle_Q = \frac{1}{\sqrt{\exp_Q(|z|^2)}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{[n]_Q!}} |n\rangle \tag{57}$$

which is the math-type q-deformed coherent state constructed in [35].

• On the other hand, setting $\gamma = 1$, $\mathcal{Z}_q = 1/\sqrt{\omega}$ and identifying $Q \to q^2$ in the generalized Q-deformed coherent states (56), we obtain the expression

$$|z; Q; a_s\rangle_C \to |z; q\rangle = \sqrt{(\xi_Q; q^2)_\infty} \sum_{n=0}^{\infty} \frac{\xi_Q^n}{\sqrt{\omega^n [n]_{q^2}!}} |n\rangle \tag{58}$$

with $[n]_{q^2}! = [n]_{q^2}[n-1]_{q^2} \ldots, [2]_{q^2}[1]_{q^2}$ and $[n]_{q^2} \equiv (1-q^{2n})/(1-q^2)$, which is the coherent state $|z; q\rangle$ built by Spiridonov [36] for a quantum deformed harmonic oscillator potential. In this case, the parameter ω is related to the q-deformed commutation relation of the ladder operators $\hat{a}\hat{a}^\dagger - q^2\hat{a}^\dagger\hat{a} = \omega$.

5.1.3. Maths-type q-deformed coherent states ($X = D$). From (47) and (33) we obtain $\Theta_{nk}^{(D)} = q^{\gamma(n-k-1)}[\gamma(n-k)]_q$. Using this result and the constant value $\mathcal{Z}_s^{(q)} = \mathcal{Z}_q$ in (37), we find

$$h_n^{(D)}(q; a_s) = \sqrt{\frac{q^{n(1-\gamma)}(q^{2\gamma}; q^{2\gamma})_n}{(1-q^2)^n \mathcal{Z}_q^{2n}}}. \tag{59}$$

Using this result in (36) we obtain the normalized coherent state expression

$$|z; q; a_s\rangle_D = \frac{1}{\sqrt{e_{q^{2\gamma}}(|\chi_q|^2)}} \sum_{n=0}^{\infty} \frac{\chi_q^n}{\sqrt{(q^{2\gamma}; q^{2\gamma})_n}} |n\rangle \qquad \text{where} \qquad \chi_q = \sqrt{q^{(\gamma-1)}(1-q^2)} z \mathcal{Z}_q. \tag{60}$$

In this case the q^μ extension of the q-exponential function still given by expression (51) changing $Q \to q$.

As in the previous quantum deformed models next we consider some remarks about our general results.

• Hence in the limit $q \to 1$ the coherent state (60) reproduce the generalized expression obtained in [22] for the coherent state of a non-deformed harmonic oscillator system, as in the previous examples.

• Note that the math-type q-deformed coherent state $|z\rangle_q$ of [35] can be reproduced setting $\gamma = \frac{1}{2}$ and $\mathcal{Z}_q = \sqrt{\sqrt{q}/(1+q)}$ in (60).

• Another thing to observe is that, as in the previous q-deformed model, with the same choice of $\gamma = 1$ and $\mathcal{Z}_q = 1/\sqrt{\omega}$ in (60) we reproduce the coherent state $|z; q\rangle$ presented by Spiridonov [36] for a quantum deformed harmonic oscillator potential.

Finally, in order to conclude these examples we must observe two aspects: (a) the results found in the literature for harmonic oscillator q-deformed coherent states can be reproduced as particular cases of our generalized formalism; (b) it is not possible to define a q-deformed coherent state for $X = S$ deformation model since for this potential $R(a_0) = R(a_1) = \cdots =$ etc, and thus the shape invariance preservation condition (24) cannot be satisfied.

5.2. Quantum deformed Pöschl–Teller coherent states

The Pöschl–Teller potential [37], originally introduced in a molecular physics context, is closely related to several other potentials widely used in molecular and solid state physics. Besides that, the Pöschl–Teller potential presents the interesting property of represents the infinite square-well as a special limit. The partners potentials $V_\pm(x)$ for this system [18] are obtained with the superpotentials $W(x, a_1) = \sqrt{\hbar\Omega}\,\{\beta(a_1+\gamma)\cot[\beta(x+\lambda)] + \delta\csc[\beta(x+\lambda)]\}$, where β, γ, δ and λ are real constants while the remainders in the shape invariance condition (2) are given by $R(a_1) = \beta^2\eta\,[2(a_1+\gamma)+\eta]$ and the potential parameters are related by $a_{n+1} = a_n + \eta$, with $\eta = \sqrt{\hbar/(2M\Omega)}$. Using these facts we find that

$$e_n = \kappa^2 n(n+2\rho), \qquad \text{where} \qquad \kappa = \eta\beta, \qquad \rho = (a_1+\gamma)/\eta. \tag{61}$$

5.2.1. Standard q-deformed coherent states $(X = B)$. In this case with (61) in (33) we find $\Theta_{nk}^{(B)} = [\kappa^2\{n(n+2\rho) - k(k+2\rho)\}]_q$ and substituting this result in (37) we can show that

$$\prod_{k=0}^{n-1}\sqrt{\Theta_{nk}^{(B)}} = \frac{q^{\frac{1}{3}\kappa^2 n(n+1)[n+\frac{3}{2}(\rho-\frac{1}{6})]}}{(q-q^{-1})^{n/2}}\sqrt{(2\kappa^2, 2\rho; q)_n} \tag{62}$$

where we used the recurrence relation $R(a_k) = R(a_1) + 2\kappa^2(k-1)$ and introduced the two-parameters generalization of q-shifted factorial defined as $(\mu, a; q)_0 = 1$ and

$$(\mu, a; q)_n = \prod_{k=0}^{n-1}(1 - q^{\mu[k(k+a)-n(n+a)]}), \qquad \text{with} \quad n = 1, 2, 3, \ldots. \tag{63}$$

In order to explore our general approach in the construction of q-deformed coherent states let us to introduce some forms for the arbitrary functional $\mathcal{Z}_s^{(q)} \equiv \mathcal{Z}^{(q)}(a_s)$ and verify their consequences. First, assuming the constant value $\mathcal{Z}_s^{(q)} = \mathcal{Z}_q$ and using it and (62) in (37), we find for the coherent state (36)

$$|z; q; a_s\rangle_B = \frac{1}{\sqrt{\mathcal{E}_{1,1}^{(q)}(2\kappa^2, 2\rho; |\xi_q|^2)}}\sum_{n=0}^{\infty}\left\{\frac{q^{-\frac{1}{3}\kappa^2 n^2[n+\frac{3}{4}(2\rho+1)]}}{\sqrt{(2\kappa^2, 2\rho; q)_n}}\right\}\xi_q^n|n\rangle \qquad \text{where}$$
$$\xi_q = \sqrt{\frac{q-q^{-1}}{q^{\kappa^2(\rho-\frac{1}{6})}}}\,z\mathcal{Z}_q \tag{64}$$

and the q-deformed function $\mathcal{E}_{\mu,\sigma}^{(q)}(a, b; z)$ is defined as

$$\mathcal{E}_{\mu,\sigma}^{(q)}(a, b; z) = \sum_{n=0}^{\infty}\left\{\frac{q^{-\frac{\mu}{3}an^2[n+\frac{3}{4}(b+\sigma)]}}{(a, b; q)_n}\right\}z^n. \tag{65}$$

Note that when $\mathcal{Z}_q = \kappa$, if we take the limit $\lim_{q\to1}\left\{h_n^{(B)}(q; a_s)\right\} = \sqrt{\Gamma(n+1)\Gamma(2\rho+2n)/\Gamma(2\rho+n)}$ and use this result in (64), we obtain

$$\lim_{q\to1}|z; q; a_s\rangle_B = |z; a_s\rangle = \left[\sum_{n=0}^{\infty}\frac{\Gamma(2\rho+n)}{\Gamma(n+1)\Gamma(2\rho+2n)}|z|^{2n}\right]^{-1/2}$$
$$\times\sum_{n=0}^{\infty}\sqrt{\frac{\Gamma(2\rho+n)}{\Gamma(2\rho+2n)\Gamma(n+1)}}\,z^n|n\rangle. \tag{66}$$

Equation (66) is the generalized coherent state for non-deformed Pöschl–Teller potential obtained in [22]. In this sense (64) can be assumed as the q-deformed version of the generalized coherent state obtained in that reference.

On the other hand, assuming $\rho = \frac{1}{2}$, $\mathcal{Z}_q = \kappa$ and taking the limit $q \to 1$ it is possible to get

$$\lim_{q\to 1} |z; q; a_s\rangle_B = |z; a_s\rangle = \sqrt{\mathrm{sech}(|z|)} \sum_{n=0}^{\infty} \left\{ \frac{z^n}{\sqrt{(2n)!}} \right\} |n\rangle \tag{67}$$

which is the simple expression found by Fukui in [14] for the Pöschl–Teller coherent states.

At this stage let us define an auxiliary function linear in the a_n-parameter $g_s \equiv g(c, d; a_s) = ca_s + d$ and its q-number version $g_s^{(q)} \equiv g^{(q)}(c, d; a_s) = [ca_s + d]_q$, where c and d are constants. With the help of the a_n-potential parameters relation we can show that

$$\prod_{k=0}^{n-1} g_{s+k}^{(q)} = \left(\frac{q^{-g_s}}{q^{-1} - q} \right)^n q^{-\frac{1}{2}n(n-1)c\eta} (q^{2g_s}; q^{2c\eta})_n. \tag{68}$$

Now, as a second possibility, if we define the functional $\mathcal{Z}_s^{(q)} = \sqrt{g^{(q)}(2\kappa/\eta, \kappa; a_1) g^{(q)}(2\kappa/\eta, 2\kappa; a_1)}$ and use (68) we obtain

$$\prod_{k=0}^{n-1} \mathcal{Z}_{s+k}^{(q)} = \left(\frac{q^{-\kappa(\nu+\frac{1}{2})}}{q^{-1} - q} \right)^n q^{-\kappa n^2} \sqrt{(q^{2\kappa(\nu+1)}; q^{4\kappa})_n (q^{2\kappa(\nu+2)}; q^{4\kappa})_n} \tag{69}$$

where $\nu = 2a_1/\eta$. Inserting equations (62) and (69) in (37) we obtain

$$h_n^{(B)}(q; a_s) = (q - q^{-1})^{\frac{n}{2}} q^{\phi(n)} \sqrt{\frac{(2\kappa^2, 2\rho; q)_n}{(q^{2\kappa(\nu+1)}; q^{4\kappa})_n (q^{2\kappa(\nu+2)}; q^{4\kappa})_n}} \tag{70}$$

where $\phi(n) = \kappa n^2 \left\{ \frac{\kappa}{3} \left[n + \frac{3}{4}(2\rho + 1) \right] + 1 \right\} + \kappa n \left[\frac{\kappa}{2} \left(\rho - \frac{1}{6} \right) + \nu + \frac{1}{2} \right]$. Therefore the coherent state (36) obtained with these results is

$$\begin{aligned} |z; q; a_s\rangle_B = {} & \frac{1}{\sqrt{\mathbb{F}_{1,1}^{(q)}(2\kappa, 2\rho, \nu; |\xi_q|^2)}} \sum_{n=0}^{\infty} q^{-\kappa n^2 \{\frac{1}{3}\kappa[n + \frac{3}{4}(2\rho+1)]+1\}} \\ & \times \sqrt{\frac{(q^{2\kappa(\nu+1)}; q^{4\kappa})_n (q^{2\kappa(\nu+2)}; q^{4\kappa})_n}{(2\kappa^2, 2\rho; q)_n}} \xi_q^n |\Psi_n\rangle \end{aligned} \tag{71}$$

where $\xi_q = q^{-\frac{1}{4}\kappa[\kappa(\rho-\frac{1}{6})+2\nu+1]} z / \sqrt{q - q^{-1}}$ and the q-deformed function $\mathbb{F}_{\mu,\sigma}^{(q)}(a, b, c; z)$ is defined as

$$\mathbb{F}_{\mu,\sigma}^{(q)}(a, b, c; z) = \sum_{n=0}^{\infty} q^{-an^2\{\frac{1}{6}\mu a[n+\frac{3}{4}(b+\sigma)]+1\}} \left[\frac{(q^{a(c+1)}; q^{2a})_n (q^{a(c+2)}; q^{2a})_n}{(\frac{1}{2}a^2, b; q)_n} \right] z^n. \tag{72}$$

Note that if we assume $\gamma = \eta/2$, take the limit $\lim_{q\to 1} \left\{ h_n^{(B)}(q; a_s) \right\} = \sqrt{\Gamma(\nu + 1)\Gamma(n + 1)/\Gamma(\nu + n + 1)}$ and use this result in (71) we obtain

$$\lim_{q\to 1} |z; q; a_s\rangle_B = \{1 - |z|^2\}^{\frac{1}{2}(\nu+1)} \sum_{n=0}^{\infty} \sqrt{\frac{\Gamma(\nu + n + 1)}{\Gamma(\nu + 1)\Gamma(n + 1)}} z^n |\Psi_n\rangle. \tag{73}$$

Looking at (73) we recognize the coherent state of the Pöschl–Teller potential of first type obtained in [38].

5.2.2. Standard Q-deformed coherent states $(X = C)$. In this case with (61) in (33) we find $\Theta_{nk}^{(C)} = [\kappa^2\{n(n+2\rho) - k(k+2\rho)\}]_Q$ and substituting this result in (37) we can show that

$$\prod_{k=0}^{n-1}\sqrt{\Theta_{nk}^{(C)}} = \frac{Q^{\kappa^2 n\{\frac{2}{3}n[n+\frac{3}{4}(2\rho+1)]+\rho-\frac{1}{6}\}}}{\{Q-1\}^{\frac{n}{2}}}\sqrt{(\kappa^2, 2\rho; Q)_n} \tag{74}$$

where we defined the Q version $(\mu, a; Q)_n$ of the two-parameters generalization of q-shifted factorial, obtained with the replacement of $q \to Q$ in (63). With the simple choice $\mathcal{Z}_s^{(q)} = \mathcal{Z}_Q$, a constant, in (74) and (37) we find for the coherent state (36)

$$|z; q; a_s\rangle_C = \frac{1}{\sqrt{\mathcal{E}_{4,1}^{(Q)}(\kappa^2, 2\rho; |\xi_Q|^2)}}\sum_{n=0}^{\infty}\left\{\frac{Q^{-\frac{2}{3}\kappa^2 n^2[n+\frac{3}{4}(2\rho+1)]}}{\sqrt{\left(\kappa^2, 2\rho; Q\right)_n}}\right\}\xi_Q^n|\Psi_n\rangle \qquad \text{where}$$

$$\xi_Q = \frac{\sqrt{Q-1}\,z\mathcal{Z}_Q}{Q^{\kappa^2(\rho-\frac{1}{6})}} \tag{75}$$

and with the Q-deformed function $\mathcal{E}_{\mu,\sigma}^{(Q)}(a, b; z)$ defined as (65) just with the replacement of $q \to Q$.

It is worth emphasizing that, as in the $X = B$ deformation model, from (75) it is possible to get the results obtained in [22] and [14] in the limit of $Q \to 1$ using the same conditions assumed in those cases.

By using the same auxiliary linear function $g(c, d; a_s)$ and introducing its Q-number version $g_s^{(Q)} \equiv g^{(Q)}(c, d; a_s) = [ca_s + d]_Q$, it is possible to verify that in this case

$$\prod_{k=0}^{n-1} g_{s+k}^{(Q)} = \frac{\sqrt{(Q^{g_s}; Q^{c\eta})_n}}{\{1-Q\}^{\frac{n}{2}}}. \tag{76}$$

As a second choice, if we define the Q-version $\mathcal{Z}_s^{(Q)}$ of the functional $\mathcal{Z}_s^{(q)}$ introduced in the previous case we obtain

$$\prod_{k=0}^{n-1}\mathcal{Z}_{s+k}^{(Q)} = \frac{\sqrt{(Q^{\kappa(\nu+1)}; Q^{2\kappa})_n (Q^{\kappa(\nu+2)}; Q^{2\kappa})_n}}{(1-Q)^n}. \tag{77}$$

Thus inserting equations (74) and (77) in (37) we obtain

$$h_n^{(C)}(Q; a_s) = (-i)^n\{1-Q\}^{\frac{n}{2}}Q^{\phi(n)}\sqrt{\frac{(\kappa^2, 2\rho; Q)_n}{(Q^{\kappa(\nu+1)}; Q^{2\kappa})_n (Q^{\kappa(\nu+2)}; Q^{2\kappa})_n}} \tag{78}$$

where $\phi(n) = \frac{2}{3}\kappa^2 n\left\{n\left[n+\frac{3}{4}(2\rho+1)\right]+\rho-\frac{1}{6}\right\}$. The coherent state (36) obtained with these results is

$$|z; Q; a_s\rangle_C = \frac{1}{\sqrt{\mathbb{F}_{4,1}^{(Q)}(\kappa, 2\rho, \nu; |\xi_Q|^2)}}\sum_{n=0}^{\infty} Q^{-\frac{2}{3}\kappa^2 n^2[n+\frac{3}{4}(2\rho+1)]}$$

$$\times\sqrt{\frac{(Q^{\kappa(\nu+1)}; Q^{2\kappa})_n (Q^{\kappa(\nu+2)}; Q^{2\kappa})_n}{(\kappa^2, 2\rho; Q)_n}}\,\xi_Q^n|\Psi_n\rangle \tag{79}$$

where $\xi_Q = \mathrm{i}Q^{-\frac{2}{3}\kappa^2(\rho-\frac{1}{6})}z/\sqrt{1-Q}$ and the Q-deformed function $\mathbb{F}_{\mu,\sigma}^{(Q)}(a, b, c; z)$ is defined as

$$\mathbb{F}_{\mu,\sigma}^{(Q)}(a, b, c; z) = \sum_{n=0}^{\infty} Q^{-\frac{1}{3}\mu a^2 n^2[n+\frac{3}{4}(b+\sigma)]}\left[\frac{(Q^{a(c+1)}; Q^{2a})_n (Q^{a(c+2)}; Q^{2a})_n}{(a^2, b; Q)_n}\right]z^n. \tag{80}$$

It should be noted that the coherent state of the Pöschl–Teller potential of first type obtained in [37] can be reproduced from (79) assuming $\gamma = \eta/2$ and taking the limit of the expansion factor (78) when Q goes to unity.

5.2.3. Maths-type q-deformed coherent states ($X = D$). In this case $\Theta_{nk}^{(D)} = q^{\kappa^2[n(n+2\rho)-(k+1)(k+2\rho+1)]}[\kappa^2\{n(n+2\rho)-k(k+2\rho)\}]_q$ and with this result in (37) we get

$$\prod_{k=0}^{n-1}\sqrt{\Theta_{nk}^{(D)}} = \frac{q^{\frac{2}{3}\kappa^2 n[n(n+\frac{3}{2}\rho)-\frac{1}{4}]}}{\{q-q^{-1}\}^{\frac{n}{2}}}\sqrt{(2\kappa^2, 2\rho; q)_n}. \tag{81}$$

The coherent state (36) obtained with this result and the simple choice $\mathcal{Z}_s^{(q)} = \mathcal{Z}_q$ will be the form

$$|z;q;a_s\rangle_D = \frac{1}{\sqrt{\mathcal{E}_{2,0}^{(q)}(2\kappa^2, 2\rho; |\xi_q|^2)}}\sum_{n=0}^{\infty}\left\{\frac{q^{-\frac{2}{3}\kappa^2 n^2(n+\frac{3}{2}\rho)}}{\sqrt{(2\kappa^2,2\rho;q)_n}}\right\}\xi_q^n|\Psi_n\rangle \qquad \text{where}$$

$$\xi_q = q^{\frac{1}{6}\kappa^2}\sqrt{q-q^{-1}}\,z\mathcal{Z}_q. \tag{82}$$

On the other hand, with the same form of the previous model ($X = C$) for $\mathcal{Z}_s^{(q)}$ we find an expression for $h_n^{(D)}$ with the same form presented by equation (70) but with q power argument now given by $\phi(n) = \kappa n^2\left[\frac{2}{3}\kappa\left(n+\frac{3}{2}\rho\right)+1\right]+\kappa n\left(\nu+\frac{1}{2}-\frac{1}{6}\kappa\right)$. The coherent state (36) obtained in these circumstances has the form

$$|z;q;a_s\rangle_D = \frac{1}{\sqrt{\mathbb{F}_{4,0}^{(q)}(2\kappa, 2\rho, \nu; |\xi_q|^2)}}\sum_{n=0}^{\infty} q^{-\kappa n^2[\frac{2}{3}\kappa(n+\frac{3}{2}\rho)+1]}$$

$$\times\sqrt{\frac{(q^{2\kappa(\nu+1)};q^{4\kappa})_n(q^{2\kappa(\nu+2)};q^{4\kappa})_n}{(2\kappa^2,2\rho;q)_n}}\,\xi_q^n|\Psi_n\rangle \tag{83}$$

where $\xi_q = q^{-\frac{1}{4}\kappa[2\nu+1-\frac{1}{3}\kappa]}z/\sqrt{q-q^{-1}}$.

5.2.4. Shape invariant q-deformed coherent states ($X = S$). Since for this potential the remainders are related by $R(a_n) = \beta^2\left(a_{n+1}^2 - a_n^2\right)$ it is easy to verify that if we define the preservation functional as $\mathcal{F}_q = q^{\beta^2 a_0^2}$ then condition (24) is satisfied because

$$q^{2R(a_0)}\mathcal{F}_q^2 = q^{2\beta^2(a_1^2-a_0^2)}q^{2\beta^2 a_0^2} = q^{2\beta^2 a_1^2} = \hat{T}(a_1)\mathcal{F}_q^2\hat{T}^\dagger(a_1). \tag{84}$$

With $\mathcal{F}_q$ in (33) we get $\Theta_{nk}^{(S)} = q^{2\beta^2 a_1^2}q^{(e_n+e_k)}[e_n-e_k]_q$ and using this result and (61) in (37) we can show that

$$\prod_{k=0}^{n-1}\sqrt{\Theta_{nk}^{(S)}} = \frac{q^{\kappa^2 n^2(n+2\rho)}q^{n\beta^2 a_1^2}}{\{q-q^{-1}\}^{\frac{n}{2}}}\sqrt{(2\kappa^2,2\rho;q)_n}. \tag{85}$$

The coherent state (36) obtained with this result and the simple choice $\mathcal{Z}_s^{(q)} = \mathcal{Z}_q$ has the form

$$|z;q;a_s\rangle_S = \frac{1}{\sqrt{\mathcal{E}_{3,2\rho/3}^{(q)}(2\kappa^2,2\rho;|\xi_q|^2)}}\sum_{n=0}^{\infty}\left\{\frac{q^{-\kappa^2 n^2(n+2\rho)}}{\sqrt{(2\kappa^2,2\rho;q)_n}}\right\}\xi_q^n|\Psi_n\rangle \qquad \text{where}$$

$$\xi_q = q^{-\beta^2 a_1^2}\sqrt{q-q^{-1}}\,z\mathcal{Z}_q. \tag{86}$$

As in the previous case, with the form defined in the $X = C$ case for $\mathcal{Z}_s^{(q)}$ we find an expression for $h_n^{(D)}$ with the same form presented by equation (70) but with q power argument now given by $\phi(n) = \kappa n^2[\kappa(n+2\rho)+1]+\kappa n\left(\nu+\frac{1}{2}+\beta^2 a_1^2\right)$. Therefore the coherent state

(36) obtained in this case is given by

$$|z; q; a_s\rangle_S = \frac{1}{\sqrt{\mathbb{F}^{(q)}_{3,2\rho/3}(2\kappa, 2\rho, \nu; |\xi_q|^2)}} \sum_{n=0}^{\infty} q^{-\kappa n^2[\kappa(n+2\rho)+1]} \times \sqrt{\frac{(q^{2\kappa(\nu+1)}; q^{4\kappa})_n (q^{2\kappa(\nu+2)}; q^{4\kappa})_n}{(2\kappa^2, 2\rho; q)_n}} \xi_q^n |\Psi_n\rangle \tag{87}$$

where $\xi_q = q^{-[\frac{1}{2}\kappa(2\nu+1)+\beta^2 a_1^2]} z/\sqrt{q - q^{-1}}$.

6. Conclusions and final remarks

In this paper, using an algebraic approach, we constructed generalized coherent states for primary shape-invariant systems quantum deformed with four different models. This generalization based on the introduction of a factor functional $\mathcal{Z}^{(q)}(a_s)$ of the potential parameters in the coherent state definition (a) satisfies the set of essential requirements we enumerated in the introduction to establish classical and quantum correspondence, (b) reproduce, as particular cases, the results already known for q-deformed harmonic oscillator, the only field explored until now in the construction of quantum deformed coherent states and (c) gives several possible expressions for q-deformed coherent states obtained from primary shape-invariant systems.

It should be noted that the known q-deformed functions are usually related with harmonic oscillator quantum deformed models, where the factor e_n has a linear dependence on n. However, the Pöschl–Teller potential, like most other shape invariant potentials, has a nonlinear dependence on the quantum number n in its eigenvalue factor e_n. This fact requires the introduction of additional q-deformed functions in the expressions of the generalized coherent states for these potentials. In these new functions the deformation parameter q appears in the power of quadratic functions in n. Since $q < 1$ there are no problems with the convergence of the series representing these functions.

Another important ingredient of the generalized formalism developed here is the freedom in the construction of q-deformed coherent states for primary shape-invariant systems. In our applications we used very simple forms for the arbitrary functional $\mathcal{Z}^{(q)}(a_s)$ while the formalism permits the use of other expressions. Certainly this fact is a very relevant aspect to be considered in the applications of the coherent states. On the other hand, the study of quantum deformed extension of the shape invariant systems other than the harmonic oscillator, as well as coherent states for these systems, is a very recent, and consequently, is an open field to be explored further.

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